國立清華大學數學系碩士班純數組

碩士論文

某些纖維的 Switzer 公式

The Switzer's Formula for Certain Fibre

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Abstarct

We use the Switzer's formula to compute the coaction map for the fiber of the Kahn-Priddy map.

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Contents

1. Introduction

Consider the Kahn-Priddy map $\lambda : \Sigma^n RP^{n-1} \longrightarrow S^n$ which is the adjoint of $RP^{n-1} \xrightarrow{i_n} O(n) \xrightarrow{j_n} \Omega^n S^n$, where i_n represents a line L through the origin in R^n as the reflection in the hyperplane perpendicular to L, and j_n represents an element of $O(n)$ as a map $(R^n \cup \infty, \infty) \longrightarrow (R^n \cup \infty, \infty)$.

Let F be the fibre of $RP^\infty \xrightarrow{\lambda} S^0$. So we have the cofibration

$$S^{-1} \longrightarrow F \longrightarrow RP^\infty.$$

Since $\lambda_* : \widetilde{H}_*(RP^\infty) \longrightarrow \widetilde{H}_*(S^0)$ is a zero map in mod 2 coefficient, hence we have a short exact sequence for mod 2 homology

$$0 \longrightarrow \widetilde{H}_*(S^{-1}) \longrightarrow \widetilde{H}_*(F) \longrightarrow \widetilde{H}_*(RP^\infty) \longrightarrow 0.$$

Therefore, as a Z_2-vector space $\widetilde{H}_*(F)$ is isomorphic to $\widetilde{H}_*(S^{-1}) \oplus \widetilde{H}_*(RP^\infty)$. Let b_i be the generator of $\widetilde{H}_i(RP^\infty)$ for each $i \geq 1$ which is the dual of w_1^i, where w_1 is the generator of $\widetilde{H}^1(RP^\infty)$, and b_{-1} be the generator of

$$\widetilde{H}_*(S^{-1}) = \begin{cases} Z_2 & \text{if } * = -1 \\ 0 & \text{if } * \neq -1 \end{cases},$$

which is the dual of w_{-1}, where w_{-1} is the generator of $\widetilde{H}^*(S^{-1})$,then $\widetilde{H}_*(F)$ is generated by b_i and b_{-1}, $i \geq 1$.

Let Y be a topological space, $A_* \cong Z_2[\xi_1, \xi_2, ...]$ be the dual of the mod 2 Steenrod algebra A with $\xi_i \in A_{2^i-1}$. Then the mod 2 reduced homology $\widetilde{H}_*(Y)$ is a comodule over A_*; that is, there is a coaction

$$\mu_* : \widetilde{H}_*(Y) \longrightarrow A_* \otimes \widetilde{H}_*(Y).$$

μ_* is the dual of the action map

$$A \otimes \widetilde{H}^*(Y) \longrightarrow \widetilde{H}^*(Y).$$

Let X be the formal sum $1 + \xi_1 + \xi_2 + \cdots$, then we have the following theorem.

Theorem 1. Consider the coaction map $\mu_* : \widetilde{H}_*(F) \longrightarrow A_* \otimes \widetilde{H}_*(F)$. Then for $i \geqslant 1$,

$$\mu_*(b_i) = \sum_{k=0}^{i} (X^k)_{i-k} \otimes b_k + (X^j)_{i+1} \otimes b_{-1},$$

and

$$\mu_*(b_{-1}) = 1 \otimes b_{-1},$$

where $j = 2^m - 1$ if $2^{m-1} < i + 1 < 2^m$ for some integer $m \geqslant 1$, or $j = i + 1$ if $i + 1 = 2^m$ for some integer $m \geqslant 1$.

To prove Theorem 1, we need Theorem 2.

Theorem 2. The dual of the i-th mod 2 Steenrod square Sq^i is $(X^{2^m-1})_i$ if $2^{m-1} < i < 2^m$ for some integer $m \geqslant 1$, or $(X^i)_i$ if $i = 2^m$ for some integer $m \geqslant 1$.

To prove Theorem 2, we need a lemma.

Lemma 1. For all positive integer i, let $j = 2^m - 1$ if $2^{m-1} < i < 2^m$ for some integer $m \geqslant 1$, or $j = i$ if $i = 2^m$ for some integer $m \geqslant 1$. Then $Sq^i(w_1^j) = w_1^{i+j}$, and $Sq^I(w_1^j) = 0$, where $I = (i_1, i_2, ..., i_k)$ is any admissible sequence with $i_1 + i_2 + ... + i_k = i$, $k > 1$.

2. Calculation

We can use the multiplication map

$$\varphi : A \otimes A \longrightarrow A,$$

and it's dual

$$\varphi^* : A_* \longrightarrow A_* \otimes A_*$$

to calculate the dual of Sq^i for each i, where $\varphi(Sq^I \otimes Sq^J) = Sq^I Sq^J$ for any admissible sequence I, J and $\varphi^*(\xi_i) = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k$.

Now we calculate the case for $i = 1, 2, 3, ..., 11$.

For $i = 1, 2$, since the only element with degree 1 in A_* is ξ_1 and the only element with degree 2 in A_* is ξ_1^2, $Sq_*^1 = \xi_1$ and $Sq_*^2 = \xi_1^2$.

For $i = 3$, since
$\varphi(Sq^3 \otimes 1) = Sq^3$,
$\varphi(Sq^2 \otimes Sq^1) = Sq^2 Sq^1$,
$\varphi(Sq^1 \otimes Sq^2) = Sq^1 Sq^2 = \binom{1}{1} Sq^3 Sq^0 = Sq^3$,
$\varphi(1 \otimes Sq^3) = Sq^3$,
we have

$$\begin{aligned}\varphi^*(Sq_*^3) &= Sq_*^3 \otimes 1 + Sq_*^1 \otimes Sq_*^2 + 1 \otimes Sq_*^3 \\ &= Sq_*^3 \otimes 1 + \xi_1 \otimes \xi_1^2 + 1 \otimes Sq_*^3.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi^*(\xi_1^3 + \xi_2) &= \varphi^*(\xi_1)^3 + \varphi^*(\xi_2) = (\xi_1 \otimes 1 + 1 \otimes \xi_1)^3 + (\xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2) \\ &= (\xi_1^3 + \xi_2) \otimes 1 + \xi_1 \otimes \xi_1^2 + 1 \otimes (\xi_1^3 + \xi_2).\end{aligned}$$

Hence we have $Sq_*^3 = \xi_1^3 + \xi_2$.

For $i = 4$, since
$\varphi(Sq^4 \otimes 1) = Sq^4, \varphi(Sq^3 \otimes Sq^1) = Sq^3 Sq^1$,
$\varphi(Sq^2 Sq^1 \otimes Sq^1) = Sq^2 Sq^1 Sq^1 = Sq^2 \binom{0}{1} Sq^2 Sq^0 = 0$,
$\varphi(Sq^2 \otimes Sq^2) = Sq^2 Sq^2 = \binom{1}{2} Sq^4 + \binom{0}{0} Sq^3 Sq^1 = Sq^3 Sq^1$,
$\varphi(Sq^1 \otimes Sq^2 Sq^1) = Sq^1 Sq^2 Sq^1 = Sq^3 Sq^1$,
$\varphi(1 \otimes Sq^4) = Sq^4$,
we have

$$\varphi^*(Sq_*^4) = Sq_*^4 \otimes 1 + 1 \otimes Sq_*^4.$$

On the other hand,

$$\varphi^*(\xi_1^4) = \varphi^*(\xi_1)^4 = (\xi_1 \otimes 1 + 1 \otimes \xi_1)^4 = \xi_1^4 \otimes 1 + 1 \otimes \xi_1^4.$$

Hence we have $Sq_*^4 = \xi_1^4$.

For $i = 5$, since

$\varphi(Sq^5 \otimes 1) = Sq^5,\ \varphi(Sq^4 \otimes Sq^1) = Sq^4Sq^1,\ \varphi(Sq^3Sq^1 \otimes Sq^1) = 0,\ \varphi(Sq^3 \otimes Sq^2) = 0,$
$\varphi(Sq^2Sq^1 \otimes Sq^2) = Sq^5 + Sq^4Sq^1,\ \varphi(Sq^2 \otimes Sq^2Sq^1) = 0,\ \varphi(Sq^1 \otimes Sq^4) = Sq^5,$
$\varphi(Sq^2 \otimes Sq^3) = Sq^5 + Sq^4Sq^1,\ \varphi(Sq^1 \otimes Sq^3Sq^1) = 0,\ \varphi(1 \otimes Sq^5) = Sq^5,$

we have

$$\begin{aligned} \varphi^*(Sq_*^5) &= Sq_*^5 \otimes 1 + (Sq^2Sq^1)_* \otimes Sq_*^2 + Sq_*^2 \otimes Sq_*^3 + Sq_*^1 \otimes Sq_*^4 + 1 \otimes Sq_*^5 \\ &= Sq_*^5 \otimes 1 + \xi_2 \otimes \xi_1^2 + \xi_1^2 \otimes (\xi_1^3 + \xi_2) + \xi_1 \otimes \xi_1^4 + 1 \otimes Sq_*^5. \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi^*(\xi_1^5 + \xi_1^2\xi_2) &= (\xi_1 \otimes 1 + 1 \otimes \xi_1)^5 + (\xi_1 \otimes 1 + 1 \otimes \xi_1)^2(\xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2) \\ &= (\xi_1^5 + \xi_1^2\xi_2) \otimes 1 + \xi_2 \otimes \xi_1^2 + \xi_1^2 \otimes (\xi_1^3 + \xi_2) + \xi_1 \otimes \xi_1^4 + 1 \otimes (\xi_1^5 + \xi_1^2\xi_2). \end{aligned}$$

Hence we have $Sq_*^5 = \xi_1^5 + \xi_1^2\xi_2$.

For $i = 6$, since

$\varphi(Sq^6 \otimes 1) = Sq^6,\ \varphi(Sq^5 \otimes Sq^1) = Sq^5Sq^1,\ \varphi(Sq^4Sq^1 \otimes Sq^1) = 0,\ \varphi(Sq^4 \otimes Sq^2) = Sq^4Sq^2,$
$\varphi(Sq^3Sq^1 \otimes Sq^2) = Sq^5Sq^1,\ \varphi(Sq^3 \otimes Sq^3) = Sq^5Sq^1,\ \varphi(Sq^2Sq^1 \otimes Sq^3) = 0,$
$\varphi(Sq^3Sq^2 \otimes Sq^1) = 0,\ \varphi(Sq^2Sq^1 \otimes Sq^2Sq^1) = Sq^5Sq^1,\ \varphi(Sq^2 \otimes Sq^4) = Sq^6 + Sq^5Sq^1,$
$\varphi(Sq^2 \otimes Sq^3Sq^1) = Sq^5Sq^1,\ \varphi(Sq^1 \otimes Sq^5) = 0,\ \varphi(1 \otimes Sq^6) = Sq^6,$

we have

$$\begin{aligned} \varphi^*(Sq_*^6) &= Sq_*^6 \otimes 1 + Sq_*^2 \otimes Sq_*^4 + 1 \otimes Sq_*^6 \\ &= Sq_*^6 \otimes 1 + \xi_1^2 \otimes \xi_1^4 + 1 \otimes Sq_*^6. \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi^*(\xi_1^6 + \xi_2^2) &= (\xi_1 \otimes 1 + 1 \otimes \xi_1)^6 + (\xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2)^2 \\ &= (\xi_1^6 + \xi_2^2) \otimes 1 + \xi_1^2 \otimes \xi_1^4 + 1 \otimes (\xi_1^6 + \xi_2^2). \end{aligned}$$

Hence we have $Sq_*^6 = \xi_1^6 + \xi_2^2$.

For $i = 7$, since

$\varphi(Sq^7 \otimes 1) = Sq^7,\ \varphi(Sq^4Sq^2Sq^1 \otimes 1) = Sq^4Sq^2Sq^1,\ \varphi(Sq^6 \otimes Sq^1) = Sq^6Sq^1,$
$\varphi(Sq^5Sq^1 \otimes Sq^1) = 0,\ \varphi(Sq^4Sq^2 \otimes Sq^1) = Sq^4Sq^2Sq^1,\ \varphi(Sq^5 \otimes Sq^2) = Sq^5Sq^2,$
$\varphi(Sq^4Sq^1 \otimes Sq^2) = Sq^5Sq^2,\ \varphi(Sq^4 \otimes Sq^3) = Sq^5Sq^2,\ \varphi(Sq^3Sq^1 \otimes Sq^3) = 0,$
$\varphi(Sq^3Sq^1 \otimes Sq^2Sq^1) = 0,\ \varphi(Sq^4 \otimes Sq^2Sq^1) = Sq^4Sq^2Sq^1,\ \varphi(Sq^3 \otimes Sq^4) = Sq^7,$
$\varphi(Sq^2Sq^1 \otimes Sq^4) = Sq^6Sq^1,\ \varphi(Sq^3 \otimes Sq^3Sq^1) = 0,\ \varphi(Sq^2Sq^1 \otimes Sq^3Sq^1) = 0,$
$\varphi(Sq^2 \otimes Sq^5) = Sq^6Sq^1,\ \varphi(Sq^2 \otimes Sq^4Sq^1) = Sq^6Sq^1,\ \varphi(Sq^1 \otimes Sq^6) = Sq^7,$
$\varphi(1 \otimes Sq^7) = Sq^7,$

we have

$$\begin{aligned}\varphi^*(Sq_*^7) &= Sq_*^7\otimes 1 + Sq_*^3\otimes Sq_*^4 + Sq_*^1\otimes Sq_*^6 + 1\otimes Sq_*^7\\ &= Sq_*^7\otimes 1 + (\xi_1^3+\xi_2)\otimes \xi_1^4 + \xi_1\otimes(\xi_1^6+\xi_2^2) + 1\otimes Sq_*^7.\end{aligned}$$

On the other hand,

$$\begin{aligned}\varphi^*(\xi_1^7+\xi_1^4\zeta_2+\xi_1\xi_2^2+\xi_3) &= (\xi_1\otimes 1+1\otimes\xi_1)^7\\ &\quad +(\xi_1\otimes 1+1\otimes\xi_1)^4(\xi_2\otimes 1+\xi_1^2\otimes\xi_1+1\otimes\xi_2)\\ &\quad +(\xi_1\otimes 1+1\otimes\xi_1)(\xi_2\otimes 1+\xi_1^2\otimes\xi_1+1\otimes\xi_2)^2\\ &\quad +(\xi_3\otimes 1+\xi_2^2\otimes\xi_1+\xi_1^4\otimes\xi_2+1\otimes\xi_2)\\ &= (\xi_1^7+\xi_1^4\zeta_2+\xi_1\xi_2^2+\xi_3)\otimes 1+(\xi_1^3+\xi_2)\otimes\xi_1^4\\ &\quad +\xi_1\otimes(\xi_1^6+\xi_2^2)+1\otimes(\xi_1^7+\xi_1^4\zeta_2+\xi_1\xi_2^2+\xi_3).\end{aligned}$$

Hence we have $Sq_*^7=\xi_1^7+\xi_1^4\zeta_2+\xi_1\xi_2^2+\xi_3$.

For $i=8$, since
$\varphi(Sq^8\otimes 1)=Sq^8,\ \varphi(Sq^7\otimes Sq^1)=Sq^7Sq^1,\ \varphi(Sq^6Sq^1\otimes Sq^1)=0,$
$\varphi(Sq^5Sq^2\otimes Sq^1)=Sq^5Sq^2Sq^1,\ \varphi(Sq^4Sq^2Sq^1\otimes Sq^1)=0,\ \varphi(Sq^6\otimes Sq^2)=Sq^6Sq^2,$
$\varphi(Sq^4Sq^2\otimes Sq^2)=Sq^5Sq^2Sq^1,\ \varphi(Sq^5Sq^1\otimes Sq^2)=0,\ \varphi(Sq^5\otimes Sq^3)=0,$
$\varphi(Sq^4Sq^1\otimes Sq^3)=0,\ \varphi(Sq^5\otimes Sq^2Sq^1)=Sq^5Sq^2Sq^1,\ \varphi(Sq^3Sq^1\otimes Sq^4)=Sq^7Sq^1,$
$\varphi(Sq^4Sq^1\otimes Sq^2Sq^1)=Sq^5Sq^2Sq^1,\ \varphi(Sq^4\otimes Sq^3Sq^1)=Sq^5Sq^2Sq^1,$
$\varphi(Sq^4\otimes Sq^4)=Sq^7Sq^1+Sq^6Sq^2,\ \varphi(Sq^3Sq^1\otimes Sq^3Sq^1)=0,\ \varphi(Sq^3\otimes Sq^5)=Sq^7Sq^1,$
$\varphi(Sq^2Sq^1\otimes Sq^5)=0,\ \varphi(Sq^3\otimes Sq^4Sq^1)=Sq^7Sq^1,\ \varphi(Sq^2Sq^1\otimes Sq^4Sq^1)=0,$
$\varphi(Sq^2\otimes Sq^6)=0,\ \varphi(Sq^2\otimes Sq^5Sq^1)=0,\ \varphi(Sq^2\otimes Sq^4Sq^2)=Sq^6Sq^2,\ \varphi(Sq^1\otimes Sq^7)=0,$
$\varphi(Sq^1\otimes Sq^6Sq^1)=Sq^7Sq^1,\ \varphi(Sq^1\otimes Sq^5Sq^2)=0,\ \varphi(Sq^1\otimes Sq^4Sq^2Sq^1)=Sq^5Sq^2Sq^1,$
$\varphi(1\otimes Sq^8)=Sq^8,$
we have

$$\varphi^*(Sq_*^8)=Sq_*^8\otimes 1+1\otimes Sq_*^8.$$

On the other hand,

$$\begin{aligned}\varphi^*(\xi_1^8) &= (\xi_1\otimes 1+1\otimes\xi_1)^8\\ &= \xi_1^8\otimes 1+1\otimes\xi_1^8.\end{aligned}$$

Hence we have $Sq_*^8=\xi_1^8$.

For $i=9$, since

$\varphi(Sq^9\otimes 1)=Sq^9,\ \varphi(Sq^8\otimes Sq^1)=Sq^8Sq^1,\ \varphi(Sq^7Sq^1\otimes Sq^1)=0,\ \varphi(Sq^5Sq^2Sq^1\otimes Sq^1)=0,$
$\varphi(Sq^6Sq^2\otimes Sq^1)=Sq^6Sq^2Sq^1,\ \varphi(Sq^7\otimes Sq^2)=Sq^7Sq^2,\ \varphi(Sq^6Sq^1\otimes Sq^2)=Sq^6Sq^3,$
$\varphi(Sq^5Sq^2\otimes Sq^2)=0,\ \varphi(Sq^4Sq^2Sq^1\otimes Sq^2)=Sq^9+Sq^8Sq^1+Sq^7Sq^2+Sq^6Sq^2Sq^1,$
$\varphi(Sq^6\otimes Sq^3)=Sq^6Sq^3,\ \varphi(Sq^5Sq^1\otimes Sq^3)=0,\ \varphi(Sq^6\otimes Sq^2Sq^1)=Sq^6Sq^2Sq^1,$
$\varphi(Sq^4Sq^2\otimes Sq^3)=Sq^9+Sq^8Sq^1+Sq^7Sq^2+Sq^6Sq^2Sq^1,\ \varphi(Sq^5Sq^1\otimes Sq^2Sq^1)=0,$
$\varphi(Sq^4Sq^2\otimes Sq^2Sq^1)=0,\ \varphi(Sq^5\otimes Sq^4)=Sq^7Sq^2,\ \varphi(Sq^5\otimes Sq^3Sq^1)=0,\ \varphi(Sq^4Sq^1\otimes Sq^3Sq^1)=0,$
$\varphi(Sq^4Sq^1\otimes Sq^4)=Sq^9+Sq^8Sq^1+Sq^7Sq^2,\ \varphi(Sq^4\otimes Sq^5)=Sq^9+Sq^8Sq^1+Sq^7Sq^2,$
$\varphi(Sq^3Sq^1\otimes Sq^5)=0,\ \varphi(Sq^4\otimes Sq^4Sq^1)=Sq^6Sq^2Sq^1,\ \varphi(Sq^3Sq^1\otimes Sq^4Sq^1)=0,\ \varphi(Sq^3\otimes Sq^6)=0,$
$\varphi(Sq^2Sq^1\otimes Sq^6)=Sq^9+Sq^8Sq^1,\ \varphi(Sq^3\otimes Sq^5Sq^1)=0,\ \varphi(Sq^2Sq^1\otimes Sq^5Sq^1)=0,$
$\varphi(Sq^3\otimes Sq^4Sq^2)=Sq^7Sq^2,\ \varphi(Sq^2Sq^1\otimes Sq^4Sq^2)=Sq^6Sq^3,\ \varphi(Sq^2\otimes Sq^7)=Sq^9+Sq^8Sq^1,$
$\varphi(Sq^2\otimes Sq^6Sq^1)=0,\ \varphi(Sq^2\otimes Sq^5Sq^2)=Sq^6Sq^3,\ \varphi(Sq^2\otimes Sq^4Sq^2Sq^1)=Sq^6Sq^2Sq^1+Sq^5Sq^3Sq^1,$
$\varphi(Sq^1\otimes Sq^8)=Sq^9,\ \varphi(Sq^1\otimes Sq^7Sq^1)=0,\ \varphi(Sq^1\otimes Sq^6Sq^2)=Sq^7Sq^2,\ \varphi(Sq^1\otimes Sq^5Sq^2Sq^1)=0,$
$\varphi(1\otimes Sq^9)=Sq^9,$
we have

$$\begin{aligned}
\varphi^*(Sq^9_*) &= Sq^9_*\otimes 1+(Sq^4Sq^2Sq^1)_*\otimes Sq^2_*+(Sq^4Sq^2)_*\otimes Sq^3_*+(Sq^4Sq^1)_*\otimes Sq^4_*\\
&\quad +Sq^4_*\otimes Sq^5_*+(Sq^2Sq^1)_*\otimes Sq^6_*+Sq^2_*\otimes Sq^7_*+Sq^1_*\otimes Sq^8_*+1\otimes Sq^9_*\\
&= Sq^9_*\otimes 1+\xi_3\otimes\xi_1^2+(Sq^4Sq^2)_*\otimes(\xi_1^3+\xi_2)+(Sq^4Sq^1)_*\otimes\xi_1^4\\
&\quad +\xi_1^4\otimes(\xi_1^5+\xi_1^2\xi_2)+\xi_2\otimes(\xi_1^6+\xi_2^2)+\xi_1^2\otimes(\xi_1^7+\xi_1^4\zeta_2+\xi_1\xi_2^2+\xi_3)\\
&\quad +\xi_1\otimes\xi_1^8+1\otimes Sq^9_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi^*(\xi_1^9+\xi_1^6\xi_2+\xi_2^3+\xi_1^2\xi_3) &= (\xi_1\otimes 1+1\otimes\xi_1)^9\\
&\quad +(\xi_1\otimes 1+1\otimes\xi_1)^6(\xi_2\otimes 1+\xi_1^2\otimes\xi_1+1\otimes\xi_2)\\
&\quad +(\xi_2\otimes 1+\xi_1^2\otimes\xi_1+1\otimes\xi_2)^3\\
&\quad +(\xi_1\otimes 1+1\otimes\xi_1)^2(\xi_3\otimes 1+\xi_2^2\otimes\xi_1+\xi_1^4\otimes\xi_2+1\otimes\xi_2)\\
&= (\xi_1^9+\xi_1^6\xi_2+\xi_2^3+\xi_1^2\xi_3)\otimes 1+\xi_3\otimes\xi_1^2+\xi_2^2\otimes(\xi_1^3+\xi_2)+\xi_1^2\zeta_2\otimes\xi_1^4\\
&\quad +\xi_1^4\otimes(\xi_1^5+\xi_1^2\xi_2)+\xi_2\otimes(\xi_1^6+\xi_2^2)+\xi_1^2\otimes(\xi_1^7+\xi_1^4\zeta_2+\xi_1\xi_2^2+\xi_3)\\
&\quad +\xi_1\otimes\xi_1^8+1\otimes(\xi_1^9+\xi_1^6\xi_2+\xi_2^3+\xi_1^2\xi_3).
\end{aligned}$$

Hence we have $(Sq^4Sq^2)_*=\xi_2^2$, $(Sq^4Sq^1)_*=\xi_1^2\zeta_2$, $Sq^9_*=\xi_1^9+\xi_1^6\xi_2+\xi_2^3+\xi_1^2\xi_3$.

For $i=10$, since the elements in $A\otimes A$ which maps to the elements with Sq^{10} by φ are $Sq^{10}\otimes 1$, $Sq^4Sq^2\otimes Sq^4$, $Sq^4\otimes Sq^6$, $Sq^2\otimes Sq^8$, $1\otimes Sq^{10}$, we have

$$\begin{aligned}
\varphi^*(Sq^{10}_*) &= Sq^{10}_*\otimes 1+(Sq^4Sq^2)_*\otimes Sq^4_*+Sq^4_*\otimes Sq^6_*+Sq^2_*\otimes Sq^8_*+1\otimes Sq^{10}_*\\
&= Sq^{10}_*\otimes 1+\xi_2^2\otimes\xi_1^4+\xi_1^4\otimes(\xi_1^6+\xi_2^2)+\xi_1^2\otimes\xi_1^8+1\otimes Sq^{10}_*.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi^*(\xi_1^{10}+\xi_1^4\xi_2^2) &= (\xi_1\otimes 1+1\otimes\xi_1)^{10}+(\xi_1\otimes 1+1\otimes\xi_1)^4(\xi_2\otimes 1+\xi_1^2\otimes\xi_1+1\otimes\xi_2)^2\\
&= (\xi_1^{10}+\xi_1^4\xi_2^2)\otimes 1+\xi_2^2\otimes\xi_1^4+\xi_1^4\otimes(\xi_1^6+\xi_2^2)+\xi_1^2\otimes\xi_1^8+1\otimes(\xi_1^{10}+\xi_1^4\xi_2^2).
\end{aligned}$$

Hence we have $Sq_*^{10} = \xi_1^{10} + \xi_1^4\xi_2^2$.

For $i = 11$, since the elements in $A \otimes A$ which maps to the elements with Sq^{10} by φ are $Sq^{11} \otimes 1$, $Sq^5Sq^2 \otimes Sq^4$, $Sq^4Sq^1 \otimes Sq^6$, $Sq^5 \otimes Sq^6$, $Sq^4 \otimes Sq^7$, $Sq^3 \otimes Sq^8$, $Sq^1 \otimes Sq^{10}$, $1 \otimes Sq^{11}$, we have

$$\begin{aligned}
\varphi^*(Sq_*^{11}) &= Sq_*^{11} \otimes 1 + (Sq^5Sq^2)_* \otimes Sq_*^4 + (Sq^4Sq^1)_* \otimes Sq_*^6 + Sq_*^5 \otimes Sq_*^6 \\
&\quad + Sq_*^4 \otimes Sq_*^7 + Sq_*^3 \otimes Sq_*^8 + Sq_*^1 \otimes Sq_*^{10} + 1 \otimes Sq_*^{11} \\
&= Sq_*^{11} \otimes 1 + (Sq^5Sq^2)_* \otimes \xi_1^4 + \xi_1^2\zeta_2 \otimes (\xi_1^6 + \xi_2^2) + (\xi_1^5 + \xi_1^2\xi_2) \otimes (\xi_1^6 + \xi_2^2) \\
&\quad + \xi_1^4 \otimes (\xi_1^7 + \xi_1^4\zeta_2 + \xi_1\xi_2^2 + \xi_3) + (\xi_1^3 + \xi_2) \otimes \xi_1^8 + \xi_1 \otimes (\xi_1^{10} + \xi_1^4\xi_2^2) + 1 \otimes Sq_*^{11} \\
&= Sq_*^{11} \otimes 1 + (Sq^5Sq^2)_* \otimes \xi_1^4 + \xi_1^5 \otimes (\xi_1^6 + \xi_2^2) + \xi_1^4 \otimes (\xi_1^7 + \xi_1^4\zeta_2 + \xi_1\xi_2^2 + \xi_3) \\
&\quad + (\xi_1^3 + \xi_2) \otimes \xi_1^8 + \xi_1 \otimes (\xi_1^{10} + \xi_1^4\xi_2^2) + 1 \otimes Sq_*^{11}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi^*(\xi_1^{11} + \xi_1^8\xi_2 + \xi_1^5\xi_2^2 + \xi_1^4\xi_3) &= (\xi_1 \otimes 1 + 1 \otimes \xi_1)^{11} \\
&\quad + (\xi_1 \otimes 1 + 1 \otimes \xi_1)^8(\xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2) \\
&\quad + (\xi_1 \otimes 1 + 1 \otimes \xi_1)^5(\xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2)^2 \\
&\quad + (\xi_1 \otimes 1 + 1 \otimes \xi_1)^4(\xi_3 \otimes 1 + \xi_2^2 \otimes \xi_1 + \xi_1^4 \otimes \xi_2 + 1 \otimes \xi_2) \\
&= (\xi_1^{11} + \xi_1^8\xi_2 + \xi_1^5\xi_2^2 + \xi_1^4\xi_3) \otimes 1 + (\zeta_1\xi_2^2 + \zeta_3) \otimes \xi_1^4 + \xi_1^5 \otimes (\xi_1^6 + \xi_2^2) \\
&\quad + \xi_1^4 \otimes (\xi_1^7 + \xi_1^4\zeta_2 + \xi_1\xi_2^2 + \xi_3) + (\xi_1^3 + \xi_2) \otimes \xi_1^8 + \xi_1 \otimes (\xi_1^{10} + \xi_1^4\xi_2^2) \\
&\quad + 1 \otimes (\xi_1^{11} + \xi_1^8\xi_2 + \xi_1^5\xi_2^2 + \xi_1^4\xi_3).
\end{aligned}$$

Hence we have $(Sq^5Sq^2)_* = \zeta_1\xi_2^2 + \zeta_3$, $Sq_*^{11} = \xi_1^{11} + \xi_1^8\xi_2 + \xi_1^5\xi_2^2 + \xi_1^4\xi_3$.

3. Proof of Lemma 1

Lemma 1. For all positive integer i, let $j = 2^m - 1$ if $2^{m-1} < i < 2^m$ for some integer $m \geqslant 1$, or $j = i$ if $i = 2^m$ for some integer $m \geqslant 1$. Then $Sq^i(w_1^j) = w_1^{i+j}$, and $Sq^I(w_1^j) = 0$, where $I = (i_1, i_2, ..., i_k)$ is any admissible sequence with $i_1 + i_2 + ... + i_k = i$, $k > 1$.

Proof. If $i = 2^m$ for some integer $m \geqslant 1$,

$$Sq^i(w_1^j) = Sq^i(w_1^i) = \binom{i}{i} w_1^{i+i} = w_1^{i+i} = w_1^{i+j}.$$

For each l such that $0 < l = \sum_{k=0}^{m-1} l_k 2^k < 2^m$, where $0 \leqslant l_k \leqslant 1$, since for $a = \sum_{k=0}^{m} a_k 2^k$ and $b = \sum_{k=0}^{m} b_k 2^k > 0$ with $0 \leqslant a_k$, $b_k \leqslant 1$, $a_m = 1$, we have

$$\binom{a}{b} \equiv \prod_{k=0}^{m} \binom{a_k}{b_k} \pmod 2.$$

It implies

$$\binom{i}{l} \equiv [\prod_{k=0}^{m-1}\binom{0}{l_k}]\binom{1}{0} \pmod 2 \equiv 0 \pmod 2.$$

Moreover from this, we get $Sq^l(w_1^j) = Sq^l(w_1^i) = \binom{i}{l}w_1^{i+l} = 0$. So $Sq^I(w_1^j) = 0$ for any admissible sequence $I = (i_1, i_2, ..., i_n)$ with $i_1 + i_2 + ... + i_n = i$, $n > 1$.

If $2^{m-1} < i < 2^m$ for some integer $m \geqslant 1$, let $0 < l < 2^m$

$$Sq^l(w_1^j) = Sq^l(w_1^{2^m-1}) = \binom{2^m-1}{l}w_1^{l+2^m-1} = \binom{2^m-1}{l}w_1^{l+j}.$$

We may assume $l = \sum_{k=0}^{m-1} l_k 2^k$ for $0 \leqslant l_k \leqslant 1$. Since $2^m - 1 = \sum_{k=0}^{m-1} 2^k$, then

$$\binom{2^m-1}{l} \equiv \prod_{k=0}^{m-1}\binom{1}{l_k} \pmod 2 \equiv 1 \pmod 2.$$

That is $Sq^l(w_1^j) = w_1^{l+j}$, and $Sq^i(w_1^j) = w_1^{i+j}$ is the case for $l = i$.

Given an admissible sequence $I = (i_1, i_2, ..., i_n)$ with $i_1 + i_2 + ... + i_n = i$, $n > 1$. Let $I' = (i_1, i_2, ..., i_{n-2})$, then $Sq^I(w_1^j) = Sq^{I'}Sq^{i_{n-1}}Sq^{i_n}(w_1^j) = Sq^{I'}Sq^{i_{n-1}}(w_1^{i_n+j})$.

Actually, we may assume $i_n = 2^s + \sum_{k=0}^{s-1} b_k 2^k$ for $0 \leqslant b_k \leqslant 1$, $0 < s < m-1$, then $i_n + j = 2^s + \sum_{k=0}^{s-1} b_k 2^k + 2^m - 1 = 2^s + \sum_{k=0}^{s-1} b_k 2^k + 2^{m-1} + 2^{m-2} + ... + 1$

$= 2^m + \sum_{k=0}^{s-1}(b_k + 1)2^k$. Therefore

$$2^m + \sum_{k=0}^{s-1} 2^k \leqslant i_n + j \leqslant 2^m + \sum_{k=0}^{s-1} 2^{k+1} = 2^m + \sum_{k=1}^{s} 2^k.$$

Since I is admissible, we have $2^{s+1} + \sum_{k=0}^{s-1} b_k 2^{k+1} = 2i_n \leqslant i_{n-1} < i < 2^m$. So we can assume $i_{n-1} = 2^u + \sum_{k=0}^{u-1} c_k 2^k$ for $0 \leqslant c_k \leqslant 1$, $s+1 \leqslant u < m$, then we have $\binom{i_n+j}{i_{n-1}} \equiv 0 \pmod 2$ because the coefficent of $i_n + j$ with degree u is 0 and the coefficent of $i_{n-1}$with degree u is 1. So

$$Sq^I(w_1^j) = Sq^{I'}Sq^{i_{n-1}}(w_1^{i_n+j}) = Sq^{I'}(\binom{i_n+j}{i_{n-1}}w_1^{i_n+i_{n-1}+j}) = Sq^{I'}(0),$$

and this completes the lemma.■

4. Proof of Theorem 2

Theorem 2. The dual of the i-th mod 2 Steenrod square Sq^i is $(X^{2^m-1})_i$ if $2^{m-1} < i < 2^m$ for some integer $m \geqslant 1$, or $(X^i)_i$ if $i = 2^m$ for some integer $m \geqslant 1$.

Proof. For each positive integer i, let j as in Lemma 1. By Lemma 1, the only admissible sequence with $i_1 + i_2 + ... + i_k = i$ and $k \geq 1$, which sends w_1^j to w_1^{i+j}, is Sq^i. Since the dual of w_1^{i+j} is b_{i+j}, by Switzer's formula [3], $\mu_*(b_{i+j}) = \sum_{k=0}^{i+j}(X^k)_{i+j-k} \otimes b_k$, where the b_j term is $(X^j)_i \otimes b_j$ and $\mu_* : \widetilde{H}_*(RP^\infty) \longrightarrow A_* \otimes \widetilde{H}_*(RP^\infty)$ is the coaction map. Therefore the dual of Sq^i is $(X^j)_i$, and this completes the proof .■

5. List some examples

Note that the degree of ξ_i is $2^i - 1$, then we calculate the examples for the dual of Sq^i with $i = 1, 2, ...17$, and list the examples for the dual of Sq^i with $i = 18, 19, ...31$.

$$\begin{aligned}
Sq_*^1 &= (X^1)_1 = \frac{1!}{1!}\xi_1 \\
&= \xi_1 \\
Sq_*^2 &= (X^2)_2 = \frac{2!}{2!}\xi_1^2 \\
&= \xi_1^2 \\
Sq_*^3 &= (X^3)_3 = \frac{3!}{3!}\xi_1^3 + \frac{3!}{1!2!}\xi_2 \\
&= \xi_1^3 + \xi_2 \\
Sq_*^4 &= (X^4)_4 = \frac{4!}{4!}\xi_1^4 + \frac{4!}{1!3!}\xi_1\xi_2 \\
&= \xi_1^4 \\
Sq_*^5 &= (X^7)_5 = \frac{7!}{5!2!}\xi_1^5 + \frac{7!}{2!1!4!}\xi_1^2\xi_2 \\
&= \xi_1^5 + \xi_1^2\xi_2
\end{aligned}$$

$$
\begin{aligned}
Sq_*^6 &= (X^7)_6 = \frac{7!}{6!1!}\xi_1^6 + \frac{7!}{3!1!3!}\xi_1^3\xi_2 + \frac{7!}{2!5!}\xi_2^2 \\
&= \xi_1^6 + \xi_2^2 \\
Sq_*^7 &= (X^7)_7 = \frac{7!}{7!}\xi_1^7 + \frac{7!}{4!1!2!}\xi_1^4\xi_2 + \frac{7!}{1!2!4!}\xi_1\xi_2^2 + \frac{7!}{1!6!}\xi_3 \\
&= \xi_1^7 + \xi_1^4\zeta_2 + \xi_1\xi_2^2 + \xi_3 \\
Sq_*^8 &= (X^8)_8 = \frac{8!}{8!}\xi_1^8 + \frac{8!}{5!1!2!}\xi_1^5\xi_2 + \frac{8!}{2!2!4!}\xi_1^2\xi_2^2 + \frac{8!}{1!1!6!}\xi_1\xi_3 \\
&= \xi_1^8 \\
Sq_*^9 &= (X^{15})_9 = \frac{15!}{9!6!}\xi_1^9 + \frac{15!}{6!1!8!}\xi_1^6\xi_2 + \frac{15!}{3!2!10!}\xi_1^3\xi_2^2 + \frac{15!}{3!12!}\xi_2^3 + \frac{15!}{2!1!12!}\xi_1^2\xi_3 \\
&= \xi_1^9 + \xi_1^6\xi_2 + \xi_2^3 + \xi_1^2\xi_3 \\
Sq_*^{10} &= (X^{15})_{10} = \frac{15!}{10!5!}\xi_1^{10} + \frac{15!}{7!1!7!}\xi_1^7\xi_2 + \frac{15!}{4!2!9!}\xi_1^4\xi_2^2 + \frac{15!}{1!3!11!}\xi_1\xi_2^3 + \frac{15!}{3!1!11!}\xi_1^3\xi_3 + \frac{15!}{1!1!13!}\xi_2\xi_3 \\
&= \xi_1^{10} + \xi_1^4\xi_2^2 \\
Sq_*^{11} &= (X^{15})_{11} = \frac{15!}{11!4!}\xi_1^{11} + \frac{15!}{8!1!6!}\xi_1^8\xi_2 + \frac{15!}{5!2!8!}\xi_1^5\xi_2^2 + \frac{15!}{2!3!10!}\xi_1^2\xi_2^3 + \frac{15!}{4!1!10!}\xi_1^4\xi_3 \\
&\quad + \frac{15!}{1!1!1!12!}\xi_1\xi_2\xi_3 \\
&= \xi_1^{11} + \xi_1^8\xi_2 + \xi_1^5\xi_2^2 + \xi_1^4\xi_3 \\
Sq_*^{12} &= (X^{15})_{12} = \frac{15!}{12!3!}\xi_1^{12} + \frac{15!}{9!1!5!}\xi_1^9\xi_2 + \frac{15!}{6!2!7!}\xi_1^6\xi_2^2 + \frac{15!}{3!3!9!}\xi_1^3\xi_2^3 + \frac{15!}{4!11!}\xi_2^4 + \frac{15!}{5!1!9!}\xi_1^5\xi_3 \\
&\quad + \frac{15!}{2!1!1!11!}\xi_1^2\xi_2\xi_3 \\
&= \xi_1^{12} + \xi_2^4 \\
Sq_*^{13} &= (X^{15})_{13} = \frac{15!}{13!2!}\xi_1^{13} + \frac{15!}{10!1!4!}\xi_1^{10}\xi_2 + \frac{15!}{7!2!6!}\xi_1^7\xi_2^2 + \frac{15!}{4!3!8!}\xi_1^4\xi_2^3 + \frac{15!}{1!4!10!}\xi_1\xi_2^4 \\
&\quad + \frac{15!}{6!1!8!}\xi_1^6\xi_3 + \frac{15!}{3!1!1!10!}\xi_1^3\xi_2\xi_3 + \frac{15!}{2!1!12!}\xi_2^2\xi_3 \\
&= \xi_1^{13} + \xi_1^{10}\xi_2 + \xi_1^4\xi_2^3 + \xi_1\xi_2^4 + \xi_1^6\xi_3 + \xi_2^2\xi_3 \\
Sq_*^{14} &= (X^{15})_{14} = \frac{15!}{14!1!}\xi_1^{14} + \frac{15!}{11!1!3!}\xi_1^{11}\xi_2 + \frac{15!}{8!2!5!}\xi_1^8\xi_2^2 + \frac{15!}{5!3!7!}\xi_1^5\xi_2^3 + \frac{15!}{2!4!9!}\xi_1^2\xi_2^4 \\
&\quad + \frac{15!}{7!1!7!}\xi_1^7\xi_3 + \frac{15!}{4!1!1!9!}\xi_1^4\xi_2\xi_3 + \frac{15!}{1!2!1!11!}\xi_1\xi_2^2\xi_3 + \frac{15!}{2!13!}\xi_3^2 \\
&= \xi_1^{14} + \xi_1^8\xi_2^2 + \xi_1^2\xi_2^4 + \xi_3^2 \\
Sq_*^{15} &= (X^{15})_{15} = \frac{15!}{15!}\xi_1^{15} + \frac{15!}{12!1!2!}\xi_1^{12}\xi_2 + \frac{15!}{9!2!4!}\xi_1^9\xi_2^2 + \frac{15!}{6!3!6!}\xi_1^6\xi_2^3 + \frac{15!}{3!4!8!}\xi_1^3\xi_2^4 \\
&\quad + \frac{15!}{5!10!}\xi_2^5 + \frac{15!}{8!1!6!}\xi_1^8\xi_3 + \frac{15!}{5!1!1!8!}\xi_1^5\xi_2\xi_3 + \frac{15!}{2!2!1!10!}\xi_1^2\xi_2^2\xi_3 + \frac{15!}{1!2!12!}\xi_1\xi_3^2 + \frac{15!}{1!14!}\xi_4 \\
&= \xi_1^{15} + \xi_1^{12}\xi_2 + \xi_1^9\xi_2^2 + \xi_1^3\xi_2^4 + \xi_2^5 + \xi_1^8\xi_3 + \xi_1\xi_3^2 + \xi_4
\end{aligned}
$$

$$
\begin{aligned}
Sq_*^{16} &= (X^{16})_{16} = \frac{16!}{16!}\xi_1^{16} + \frac{16!}{13!1!2!}\xi_1^{13}\xi_2 + \frac{16!}{10!2!4!}\xi_1^{10}\xi_2^2 + \frac{16!}{7!3!6!}\xi_1^7\xi_2^3 + \frac{16!}{4!4!8!}\xi_1^4\xi_2^4 \\
&\quad + \frac{16!}{1!5!10!}\xi_1^1\xi_2^5 + \frac{16!}{9!1!6!}\xi_1^9\xi_3 + \frac{16!}{6!1!1!8!}\xi_1^6\xi_2\xi_3 + \frac{16!}{3!2!1!10!}\xi_1^3\xi_2^2\xi_3 + \frac{16!}{3!1!12!}\xi_2^3\xi_3 \\
&\quad + \frac{16!}{2!2!12!}\xi_1^2\xi_3^2 \\
&= \xi_1^{16} \\
Sq_*^{17} &= (X^{31})_{17} = \frac{31!}{17!14!}\xi_1^{17} + \frac{31!}{14!1!16!}\xi_1^{14}\xi_2 + \frac{31!}{11!2!18!}\xi_1^{11}\xi_2^2 + \frac{31!}{8!3!20!}\xi_1^8\xi_2^3 + \frac{31!}{5!4!22!}\xi_1^5\xi_2^4 \\
&\quad + \frac{31!}{2!5!24!}\xi_1^2\xi_2^5 + \frac{31!}{10!1!20!}\xi_1^{10}\xi_3 + \frac{31!}{7!1!1!22!}\xi_1^7\xi_2\xi_3 + \frac{31!}{4!2!1!24!}\xi_1^4\xi_2^2\xi_3 + \frac{31!}{1!3!1!26!}\xi_1\xi_2^3\xi_3 \\
&\quad + \frac{31!}{3!2!26!}\xi_1^3\xi_3^2 + \frac{31!}{1!2!28!}\xi_2\xi_3^2 + \frac{31!}{2!1!28!}\xi_1^2\xi_4 \\
&= \xi_1^{17} + \xi_1^{14}\xi_2 + \xi_1^8\xi_2^3 + \xi_1^2\xi_2^5 + \xi_1^{10}\xi_3 + \xi_1^4\xi_2^2\xi_3 + \xi_2\xi_3^2 + \xi_1^2\xi_4
\end{aligned}
$$

$$
\begin{aligned}
Sq_*^{18} &= \xi_2^6 + \xi_1^4\xi_3^2 + \xi_1^{12}\xi_2^2 + \xi_1^{18} \\
Sq_*^{19} &= \xi_2^4\xi_3 + \xi_1\xi_2^6 + \xi_1^4\xi_4 + \xi_1^5\xi_3^2 + \xi_1^{12}\xi_3 + \xi_1^{13}\xi_2^2 + \xi_1^{16}\xi_2 + \xi_1^{19} \\
Sq_*^{20} &= \xi_1^8\xi_2^4 + \xi_1^{20} \\
Sq_*^{21} &= \xi_3^3 + \xi_2^2\xi_4 + \xi_2^7 + \xi_1^2\xi_2^4\xi_3 + \xi_1^4\xi_2\xi_3^2 + \xi_1^6\xi_4 + \xi_1^8\xi_2^2\xi_3 + \xi_1^9\xi_2^4 + \xi_1^{12}\xi_2^3 + \xi_1^{14}\xi_3^1 + \xi_1^{18}\xi_2 + \xi_1^{21} \\
Sq_*^{22} &= \xi_1^8\xi_3^2 + \xi_1^{10}\xi_2^4 + \xi_1^{16}\xi_2^2 + \xi_1^{22} \\
Sq_*^{23} &= \xi_1^8\xi_4 + \xi_1^8\xi_2^5 + \xi_1^9\xi_3^2 + \xi_1^{11}\xi_2^4 + \xi_1^{16}\xi_3 + \xi_1^{17}\xi_2^2 + \xi_1^{20}\xi_2 + \xi_1^{23} \\
Sq_*^{24} &= \xi_2^8 + \xi_1^{24} \\
Sq_*^{25} &= \xi_2^6\xi_3 + \xi_1\xi_2^8 + \xi_1^4\xi_3^3 + \xi_1^4\xi_2^2\xi_4 + \xi_1^8\xi_2\xi_3^2 + \xi_1^{10}\xi_4 + \xi_1^{10}\xi_2^5 + \xi_1^{12}\xi_2^2\xi_3 + \xi_1^{16}\xi_2^3 + \xi_1^{18}\xi_3 + \xi_1^{22}\xi_2 + \xi_1^{25} \\
Sq_*^{26} &= \xi_2^4\xi_3^2 + \xi_1^2\xi_2^8 + \xi_1^8\xi_2^6 + \xi_1^{12}\xi_3^2 + \xi_1^{20}\xi_2^2 + \xi_1^{26} \\
Sq_*^{27} &= \xi_2^4\xi_4 + \xi_2^9 + \xi_1\xi_2^4\xi_3^2 + \xi_1^3\xi_2^8 + \xi_1^8\xi_2^4\xi_3 + \xi_1^9\xi_2^6 + \xi_1^{12}\xi_4 + \xi_1^{13}\xi_3^2 + \xi_1^{20}\xi_3 + \xi_1^{21}\xi_2^2 + \xi_1^{24}\xi_2 + \xi_1^{27} \\
Sq_*^{28} &= \xi_3^4 + \xi_1^4\xi_2^8 + \xi_1^{16}\xi_2^4 + \xi_1^{28} \\
Sq_*^{29} &= \xi_3^2\xi_4 + \xi_2^5\xi_3^2 + \xi_1\xi_3^4 + \xi_1^2\xi_2^4\xi_4 + \xi_1^2\xi_2^9 + \xi_1^5\xi_2^8 + \xi_1^8\xi_3^3 + \xi_1^8\xi_2^2\xi_4 + \xi_1^8\xi_2^7 + \xi_1^{10}\xi_2^4\xi_3 + \xi_1^{12}\xi_2\xi_3^2 + \xi_1^{14}\xi_4 \\
&\quad + \xi_1^{16}\xi_2^2\xi_3 + \xi_1^{17}\xi_2^4 + \xi_1^{20}\xi_2^3 + \xi_1^{22}\xi_3 + \xi_1^{26}\xi_2 + \xi_1^{29} \\
Sq_*^{30} &= \xi_4^2 + \xi_2^{10} + \xi_1^2\xi_3^4 + \xi_1^6\xi_2^8 + \xi_1^{16}\xi_3^2 + \xi_1^{18}\xi_2^4 + \xi_1^{24}\xi_2^2 + \xi_1^{30} \\
Sq_*^{31} &= \xi_5 + \xi_2\xi_3^4 + \xi_2^8\xi_3 + \xi_1\xi_4^2 + \xi_1\xi_2^{10} + \xi_1^3\xi_3^4 + \xi_1^4\xi_2^9 + \xi_1^7\xi_2^8 + \xi_1^{16}\xi_4 + \xi_1^{16}\xi_2^5 + \xi_1^{17}\xi_3^2 + \xi_1^{19}\xi_2^4 + \xi_1^{24}\xi_3 \\
&\quad + \xi_1^{25}\xi_2^2 + \xi_1^{28}\xi_2 + \xi_1^{31}
\end{aligned}
$$

6. Proof of Theorem 1

Theorem 1. Consider the coaction map $\mu_* : \widetilde{H}_*(F) \longrightarrow A_* \otimes \widetilde{H}_*(F)$. Then for $i \geqslant 1$,

$$\mu_*(b_i) = \sum_{k=0}^{i} (X^k)_{i-k} \otimes b_k + (X^j)_{i+1} \otimes b_{-1},$$

and

$$\mu_*(b_{-1}) = 1 \otimes b_{-1},$$

where $j = 2^m - 1$ if $2^{m-1} < i+1 < 2^m$ for some integer $m \geqslant 1$, or $j = i+1$ if $i+1 = 2^m$ for some integer $m \geqslant 1$.

Proof. Since $Sq^{i+1}(w_{-1}) = w_1^i$ for $i \geqslant 1$, we have $Sq^l Sq^{i+1}(w_{-1}) = Sq^l(w_1^i) = \binom{i}{l} w_1^{i+l} = 0$ for each $l \geqslant 2i+2$. By Theorem 2, the dual of Sq^{i+1} is $(X^j)_{i+1}$, and since the dual of b_i is w_1^i, the b_{-1} term of $\mu_*(b_i)$ is $(X^j)_{i+1} \otimes b_{-1}$. By Switzer's formula [3], the b_k term of $\mu_*(b_i)$ is $(X^k)_{i-k} \otimes b_k$. So for $i \geqslant 1$

$$\mu_*(b_i) = \sum_{k=0}^{i} (X^k)_{i-k} \otimes b_k + (X^j)_{i+1} \otimes b_{-1}.$$

Suppose there exists M in A_* such that one of the term of $\mu_*(b_{-1})$ is $M \otimes b_i$ for some $i \geqslant 1$. Since the degree of M is nonnegative, and the degree of b_{-1} is -1, it's impossible. Therefore $\mu_*(b_{-1}) = 1 \otimes b_{-1}$. This completes the proof .■

7. References

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